

BMO-ESTIMATION AND ALMOST EVERYWHERE EXPONENTIAL SUMMABILITY OF QUADRATIC PARTIAL SUMS OF DOUBLE FOURIER SERIES

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ABSTRACT. It is proved a *BMO*-estimation for quadratic partial sums of two-dimensional Fourier series from which it is derived an almost everywhere exponential summability of quadratic partial sums of double Fourier series.

1. INTRODUCTION

Let $\mathbb{T} := [-\pi, \pi)$ and $\mathbb{R} := (-\infty, \infty)$. We denote by $L_1(\mathbb{T})$ the class of all measurable functions f on \mathbb{R} that are 2π -periodic and satisfy

$$\|f\|_1 := \int_{\mathbb{T}} |f| < \infty.$$

The Fourier series of the function $f \in L_1(\mathbb{T})$ with respect to the trigonometric system is the series

$$(1) \quad \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{inx},$$

where

$$\widehat{f}(n) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} dx$$

are the Fourier coefficients of f .

Denote by $S_n(x, f)$ the partial sums of the Fourier series of f and let

$$\sigma_n(x, f) = \frac{1}{n+1} \sum_{k=0}^n S_k(x, f)$$

be the $(C, 1)$ means of (1). Fejér [1] proved that $\sigma_n(f)$ converges to f uniformly for any 2π -periodic continuous function. Lebesgue in [15] established almost everywhere convergence of $(C, 1)$ means if $f \in L_1(\mathbb{T})$. The strong summability problem, i.e. the convergence of the strong means

$$(2) \quad \frac{1}{n+1} \sum_{k=0}^n |S_k(x, f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0,$$

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was first considered by Hardy and Littlewood in [11]. They showed that for any $f \in L_r(\mathbb{T})$ ($1 < r < \infty$) the strong means tend to 0 a.e., if $n \rightarrow \infty$. The trigonometric Fourier series of $f \in L_1(\mathbb{T})$ is said to be (H, p) -summable at $x \in T$, if the values (2) converge to 0 as $n \rightarrow \infty$. The (H, p) -summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [17] for $p = 2$, and later by Zygmund [26] for the general case $1 \leq p < \infty$. K. I. Oskolkov in [19] proved the following

Theorem A. *Let $f \in L_1(\mathbb{T})$ and let Φ be a continuous positive convex function on $[0, +\infty)$ with $\Phi(0) = 0$ and*

$$(3) \quad \ln \Phi(t) = O(t / \ln \ln t) \quad (t \rightarrow \infty).$$

Then for almost all x

$$(4) \quad \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n \Phi(|S_k(x, f) - f(x)|) = 0.$$

It was noted in [19] that V. Totik announced the conjecture that (4) holds almost everywhere for any $f \in L_1(\mathbb{T})$, provided

$$(5) \quad \ln \Phi(t) = O(t) \quad (t \rightarrow \infty).$$

In [20] V. Rodin proved

Theorem B. *Let $f \in L_1(\mathbb{T})$. Then for any $A > 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n (\exp(A|S_k(x, f) - f(x)|) - 1) = 0$$

for a. e. $x \in \mathbb{T}$.

G. Karagulyan [12] proved that the following is true.

Theorem C. *Suppose that a continuous increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$, $\Phi(0) = 0$ satisfies the condition*

$$\limsup_{t \rightarrow +\infty} \frac{\log \Phi(t)}{t} = \infty.$$

Then there exists a function $f \in L_1(\mathbb{T})$ for which the relation

$$\limsup_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n \Phi(|S_k(x, f)|) = \infty$$

holds everywhere on \mathbb{T} .

In fact, Rodin in [20] has obtained a *BMO* estimate for the partial sums of Fourier series and his theorem stated above is obtained from that estimate by using John-Nirenberg theorem. Recall the definition of *BMO*[0, 1] space. It is the Banach space of functions $f \in L_1[0, 1]$ with the norm

$$\|f\|_{BMO} = \mathfrak{R}(f) + \left| \int_0^1 f(t) dt \right|$$

where

$$\mathfrak{R}(f) = \sup_I (|f - f_I|)_I, f_I = \frac{1}{|I|} \int_I f(t) dt$$

and the supremum is taken over all intervals $I \subset [0, 1]$ ([4], p. 224). Let $\{\xi_n : n = 0, 1, 2, \dots\}$ be an arbitrary sequence of numbers. Taking $\delta_k^n = [k/(n+1), (k+1)/(n+1)]$, we define

$$BMO[\xi_n] = \sup_{0 \leq n < \infty} \left\| \sum_{k=0}^n \xi_k \mathbb{I}_{\delta_k^n}(t) \right\|_{BMO}$$

where $\mathbb{I}_{\delta_k^n}(t)$ is the characteristic function of δ_k^n . Notice that the expressions

$$(6) \quad BMO[\tilde{S}_n(x, f)], \quad BMO[S_n(x, f)], \quad f \in L_1(\mathbb{T}^2), x \in \mathbb{T}$$

define a sublinear operators, where $\tilde{S}_n(x, f)$ is the conjugate partial sums. The following theorem is proved by Rodin in [20].

Theorem D. *The operators (6) are of weak type $(1, 1)$, i.e. the inequalities*

$$|\{x \in \mathbb{T} : BMO[S_n(x, f)] > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{T}} |f(t)| dt$$

and

$$|\{x \in \mathbb{T} : BMO[\tilde{S}_n(x, f)] > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{T}} |f(t)| dt$$

holds for any $f \in L_1(\mathbb{T})$.

In this paper we study the question of exponential summability of quadratical partial sums of double Fourier series. Let $f \in L_1(\mathbb{T}^2)$, $\mathbb{T}^2 := [-\pi, \pi) \times [-\pi, \pi)$ be a function with Fourier series

$$(7) \quad \sum_{m, n=-\infty}^{\infty} \hat{f}(m, n) e^{i(mx+ny)},$$

where

$$\hat{f}(m, n) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(x, y) e^{-i(mx+ny)} dx dy$$

are the Fourier coefficients of the function f . The rectangular partial sums of (7) are defined as follows:

$$S_{MN}(x, y, f) = \sum_{m=-M}^M \sum_{n=-N}^N \hat{f}(m, n) e^{i(mx+ny)}.$$

We denote by $L \log L(\mathbb{T}^2)$ the class of measurable functions f , with

$$\iint_{\mathbb{T}^2} |f| \log^+ |f| < \infty,$$

where $\log^+ u := \mathbb{I}_{\{1, \infty\}} \log u$. For cubic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [18] has proved, that if $f \in L \log L (\mathbb{T}^2)$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n (S_{kk}(x, y, f) - f(x, y)) = 0$$

for a. e. $(x, y) \in \mathbb{T}^2$. L. Zhizhiashvili [24] improved this result showing that class $L \log L (\mathbb{T}^2)$ can be replaced by $L_1 (\mathbb{T}^2)$.

From a result of S. Konyagin [14] it follows that for every $\varepsilon > 0$ there exists a function $f \in L \log^{1-\varepsilon} (\mathbb{T}^2)$ such that

$$(8) \quad \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n |S_{kk}(x, y, f) - f(x, y)| \neq 0 \quad \text{for a. e. } (x, y) \in \mathbb{T}^2.$$

The main result of the present paper is the following.

Theorem 1. *If $f \in L \log L (\mathbb{T}^2)$, then*

$$(9) \quad \begin{aligned} & |\{(x, y) \in \mathbb{T}^2 : BMO[S_{nn}(f, x, y)] > \lambda\}| \\ & \leq \frac{c}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f| \log^+ |f| \right) \end{aligned}$$

for any $\lambda > 0$, where c is an absolute positive constant.

The following theorem shows that the quadratic sums of two-dimensional Fourier series of a function $f \in L \log L (\mathbb{T}^2)$ are almost everywhere exponentially summable to the function f . It will be obtained from the previous theorem by using John-Nirenberg theorem.

Theorem 2. *Suppose that $f \in L \log L (\mathbb{T}^2)$. Then for any $A > 0$*

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{n=0}^m (\exp(A |S_{nn}(x, y, f) - f(x, y)|) - 1) = 0$$

for a. e. $(x, y) \in \mathbb{T}^2$.

According to a Lemma of L. D. Gogoladze [9], this theorem can be formulated in more general settings.

Theorem 3. *Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a increasing function, which is bounded in each finite interval $[0, a]$ and satisfies the conditions*

$$\lim_{u \rightarrow 0} \psi(u) = \psi(0) = 0, \quad \limsup_{u \rightarrow \infty} \frac{\log \psi(u)}{u} < \infty.$$

Then for any $f \in L \log L (\mathbb{T}^2)$ we have

$$\lim_{m \rightarrow \infty} \frac{1}{m+1} \sum_{n=0}^m \psi(|S_{nn}(x, y, f) - f(x, y)|) = 0$$

almost everywhere on \mathbb{T}^2 .

The results on strong summation of Marcinkiewicz type of Fourier series have been investigated in [2, 3, 10, 6, 7, 5, 8, 16, 23, 27, 28, 24]

2. PROOF OF THEOREMS

The notation $a \lesssim b$ in the proofs stands for $a \leq c \cdot b$, where c is an absolute constant.

Proof of Theorem 1. For the partial sums we have

$$S_{nn}(x, y, f) = \frac{1}{\pi^2} \iint_{\mathbb{T}^2} \frac{\sin(n+1/2)t \sin(n+1/2)s}{4 \sin(t/2) \sin(s/2)} f(x+t, y+s) dt ds.$$

Let us prove that $S_{n,n}$ can be replaced by

$$S_{nn}^*(x, y, f) = \frac{1}{\pi^2} \iint_{\mathbb{T}^2} \frac{\sin(n+1/2)t \sin(n+1/2)s}{ts} f(x+t, y+s) dt ds.$$

Indeed, putting

$$(10) \quad g(t) = \frac{1}{2 \sin(t/2)} - \frac{1}{t}$$

we have $|g(t)| \leq M$. It is easy to check that

$$\begin{aligned} (11) \quad & S_{nn}(x, y, f) \\ &= S_{nn}^*(x, y, f) \\ &+ \frac{1}{\pi^2} \iint_{\mathbb{T}^2} \frac{\sin(n+1/2)t}{t} g(s) f(x+t, y+s) dt ds \\ &+ \frac{1}{\pi^2} \iint_{\mathbb{T}^2} \frac{\sin(n+1/2)s}{s} g(t) f(x+t, y+s) dt ds \\ &+ \frac{1}{\pi^2} \iint_{\mathbb{T}^2} g(t) g(s) f(x+t, y+s) dt ds \\ &= S_{nn}^*(x, y, f) + S_{nn}^{(1)}(x, y, f) + S_{nn}^{(2)}(x, y, f) + S^{(3)}(x, y, f). \end{aligned}$$

We have

$$\begin{aligned} BMO[S_{nn}(x, y, f)] &\leq BMO[S_{nn}^*(x, y, f)] + BMO[S_{nn}^{(1)}(x, y, f)] \\ &\quad + BMO[S_{nn}^{(2)}(x, y, f)] + BMO[S^{(3)}(x, y, f)]. \end{aligned}$$

Notice that

$$\begin{aligned} S_{nn}^{(1)}(x, y, f) &= \frac{1}{\pi^2} \int_{\mathbb{T}} \frac{\sin(n+1/2)t}{t} \left(\int_{\mathbb{T}} g(s) f(x+t, y+s) ds \right) dt \\ &= S_n^*(x, F(\cdot, y)) \end{aligned}$$

is the one-dimensional modified partial sum of Fourier series of the function

$$F(\cdot, y) = \frac{1}{\pi} \int_{\mathbb{T}} g(s) f(\cdot, y + s) ds$$

for a fixed y . We have $F(\cdot, y) \in L_1(\mathbb{T})$ for all fixed y . Thus, using Theorem D we get

$$\begin{aligned} |\{(x, y) \in \mathbb{T}^2 : \text{BMO}[S_{nn}^{(1)}(x, y, f)] > \lambda\}| \\ \lesssim \frac{1}{\lambda} \iint_{\mathbb{T}^2} |F(t, y)| dt dy \lesssim \frac{1}{\lambda} \|f\|_1 \lesssim \frac{1}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f| \log^+ |f| \right). \end{aligned}$$

Similarly we will have

$$|\{(x, y) \in \mathbb{T}^2 : \text{BMO}[S_{nn}^{(2)}(x, y, f)] > \lambda\}| \lesssim \frac{1}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f| \log^+ |f| \right).$$

Since the $S^{(3)}(x, y, f)$ doesn't depend on n , using the definition of *BMO* space, we get

$$|\{(x, y) \in \mathbb{T}^2 : \text{BMO}[S^{(3)}(x, y, f)] > \lambda\}| \lesssim \frac{1}{\lambda} \left(1 + \iint_{\mathbb{T}^2} |f| \log^+ |f| \right).$$

Now discuss the operator

$$\begin{aligned} R_{n,m}(x, y, f) &:= \text{p.v.} \frac{1}{\pi^2} \iint_{\mathbb{T}^2} \frac{e^{i[(n+1/2)t + (m+1/2)s]}}{ts} f(x+t, y+s) dt ds \\ &= \text{p.v.} \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \frac{e^{i[(n+1/2)t + (m+1/2)s]}}{ts} f^*(x+t, y+s) dt ds, \quad x, y \in \mathbb{T}, \end{aligned}$$

where

$$f^*(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in \mathbb{T}^2, \\ 0, & \text{if } (x, y) \in \mathbb{R}^2 \setminus \mathbb{T}^2. \end{cases}$$

By virtue of the well-known results (see, for example, [26]) dealing with the conditions of existence of the two-dimensional Hilbert transformation, the function $R_{n,n}(x, y, f)$ is well-posed for almost all $(x, y) \in \mathbb{T}^2$ under the condition $f \in L \log L(\mathbb{T}^2)$.

Using the identity

$$\begin{aligned} e^{i(n+1/2)(t-s)} + e^{-i(n+1/2)(t-s)} - e^{i(n+1/2)(t+s)} - e^{-i(n+1/2)(t+s)} \\ = 4 \sin(n+1/2)t \sin(n+1/2)s \end{aligned}$$

we obtain

$$\begin{aligned} S_{nn}^*(x, y, f) \\ = 4(R_{n,-n}(x, y, f) + R_{-n,n}(x, y, f) - R_{n,n}(x, y, f) - R_{-n,-n}(x, y, f)). \end{aligned}$$

So we can discuss the operators $R_{\pm n, \pm n}(x, y, f)$ instead of $S_{nn}^*(x, y, f)$. Consideration of only $R_{n, n}(x, y, f)$ is enough. Below we will transform the two-dimensional singular operator $R_{n, n}$ into a composition of one-dimensional singular operators by the scheme given in [22].

For $f \in L \log L(\mathbb{T}^2)$ and for almost every $(x, y) \in \mathbb{T}^2$, we have

$$\begin{aligned}
 (12) \quad R_{n, n}(x, y, f) &= \text{p.v.} \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \frac{f^*(x+t, y+s)}{ts} e^{i(n+1/2)(t+s)} dt ds \\
 &= \text{p.v.} \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \frac{f^*(x+t, y+u-t)}{t(u-t)} e^{i(n+1/2)u} du dt \\
 &= \text{p.v.} \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \frac{e^{i(n+1/2)u}}{u} \frac{f^*(x+t, y+u-t)}{t} du dt \\
 &+ \text{p.v.} \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \frac{e^{i(n+1/2)u}}{u} \frac{f^*(x+u-t, y+t)}{t} du dt \\
 &= I_1 + I_2.
 \end{aligned}$$

For I_1 we can write

$$\begin{aligned}
 I_1 &= \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \frac{1}{\pi^2} \int_{\mathbb{R} \setminus (-\varepsilon_1, \varepsilon_1)} \frac{e^{i(n+1/2)u}}{u} \left(\int_{\mathbb{R} \setminus (-\varepsilon_2, \varepsilon_2)} \frac{f^*(x+t, y+u-t)}{t} dt \right) du \\
 &= \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \frac{1}{\pi} \int_{\mathbb{R} \setminus (-\varepsilon_1, \varepsilon_1)} \frac{e^{i(n+1/2)u}}{u} A_{\varepsilon_2}(x, y+u) du,
 \end{aligned}$$

where

$$A_{\varepsilon_2}(x, y) := \frac{1}{\pi} \int_{\mathbb{R} \setminus (-\varepsilon_2, \varepsilon_2)} \frac{f^*(x+t, y-t)}{t} dt.$$

Since $f \in L \log L(\mathbb{T}^2)$ and

$$\begin{aligned}
 A_{\varepsilon_2}(x, y-x) &= \frac{1}{\pi} \int_{\mathbb{R} \setminus (-\varepsilon_2, \varepsilon_2)} \frac{f^*(x+t, y-(x+t))}{t} dt \\
 &= \frac{1}{\pi} \int_{\mathbb{R} \setminus (-\varepsilon_2, \varepsilon_2)} \frac{F(x+t, y)}{t} dt,
 \end{aligned}$$

where

$$F(x, y) := f^*(x, y-x)$$

we conclude that (see [25] p. 443)

$$(13) \quad \sup_{\varepsilon_2} |A_{\varepsilon_2}(x, y)| \in L(\mathbb{T}^2).$$

Note also that for

$$A(x, y) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f^*(x+t, y-t)}{t} dt$$

by Zygmund's inequality (see [26], p. 404), we have

$$(14) \quad \iint_{\mathbb{T}^2} |A(x, y)| dx dy \lesssim (1 + \iint_{\mathbb{T}^2} |f(x, y)| \log^+ |f(x, y)| dx dy).$$

From (13), (14) and by Lebesgue's dominated convergence theorem we easily conclude that for almost every $(x, y) \in \mathbb{T}^2$,

$$\begin{aligned} (15) \quad I_1 &= \lim_{\varepsilon_1 \rightarrow 0} \left[\lim_{\varepsilon_2 \rightarrow 0} \frac{1}{\pi} \int_{\mathbb{R} \setminus (-\varepsilon_1, \varepsilon_1)} \frac{e^{i(n+1/2)u}}{u} A_{\varepsilon_2}(x, y+u) du \right] \\ &= \lim_{\varepsilon_1 \rightarrow 0} \left[\frac{1}{\pi} \int_{\mathbb{R} \setminus (-\varepsilon_1, \varepsilon_1)} \frac{e^{i(n+1/2)u}}{u} \lim_{\varepsilon_2 \rightarrow 0} A_{\varepsilon_2}(x, y+u) du \right] \\ &= \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{i(n+1/2)u}}{u} \left(\text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f^*(x+t, y+u-t)}{t} dt \right) du \\ &= \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{i(n+1/2)u}}{u} A(x, y+u) du, \end{aligned}$$

Analogously, we can prove that

$$(16) \quad I_2 = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{i(n+1/2)u}}{u} B(x+u, y) du,$$

where

$$B(x, y) := \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f^*(x-t, y+t)}{t} dt.$$

By Zygmund's inequality (see [26], p. 404), we have

$$(17) \quad \iint_{\mathbb{T}^2} |B(x, y)| dx dy \lesssim (1 + \iint_{\mathbb{T}^2} |f(x, y)| \log^+ |f(x, y)| dx dy).$$

Hence, from (12)-(17) we have

$$R_{n,n}(x, y, f) = R_n(x, A(\cdot, y)) + R_n(y, B(x, \cdot)),$$

where

$$R_n(x, f) = \text{p.v.} \int_{\mathbb{R}} \frac{e^{i(n+1/2)t}}{t} f^*(x+t) dt$$

is one-dimensional modified partial sums operator and $A, B \in L_1(\mathbb{T}^2)$. Observe that

$$\begin{aligned} R_n(x, f) &= \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{e^{i(n+1/2)t}}{2 \sin(t/2)} f(x+t) dt \\ &\quad - \frac{1}{\pi} \int_{\mathbb{R}} e^{i(n+1/2)t} g(t) f^*(x+t) dt \\ &= i \cdot S_n(x, f) - \tilde{S}_n(x, f) + \tilde{f}(x) \\ &\quad + \int_{\mathbb{R}} e^{i(n+1/2)t} g(t) f^*(x+t) dt, \end{aligned}$$

where $g(t)$ is the function from (10). By Theorem D $S_n(x, f)$ and $\tilde{S}_n(x, f)$ satisfy the BMO estimate. Using also

$$(18) \quad \left| \int_{\mathbb{R}} e^{i(n+1/2)t} g(t) f^*(x+t) dt \right| \lesssim \|f\|_{L_1(\mathbb{T}^2)}, \quad n = 1, 2, \dots$$

we get

$$|\{x \in \mathbb{T} : \text{BMO}[R_n(x, f)] > \lambda\}| \lesssim \frac{1}{\lambda} \int_{\mathbb{T}} |f|.$$

Thus, together with the bounds (14) and (17), we obtain

$$\begin{aligned} &|\{(x, y) \in \mathbb{T}^2 : \text{BMO}[R_{n,n}(x, y, f)] > \lambda\}| \\ &\leq |\{(x, y) \in \mathbb{T}^2 : \text{BMO}[R_n(x, A(\cdot, y))] > \lambda/2\}| \\ &+ |\{(x, y) \in \mathbb{T}^2 : \text{BMO}[R_n(y, B(x, \cdot))] > \lambda/2\}| \\ &\lesssim \frac{1}{\lambda} (\|A\|_{L^1(\mathbb{T}^2)} + \|B\|_{L^1(\mathbb{T}^2)}) \lesssim \left(1 + \iint_{\mathbb{T}^2} |f| \log^+ |f| \right) \end{aligned}$$

□

Let L_M be the Orlicz space [13] generalized by Yang function M , i. e. M is convex continuous even function such that $M(0) = 0$ and

$$\lim_{t \rightarrow 0+} \frac{M(t)}{t} = \lim_{t \rightarrow \infty} \frac{t}{M(t)} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_M} := \inf \left\{ \lambda : \lambda > 0, \int_{\mathbb{T}} M\left(\frac{|f|}{\lambda}\right) \leq 1 \right\} < \infty.$$

It is well known that L_M is a Banach space. We will need some basic properties of Orlicz spaces.

1) According to a theorem from ([13], p. 97) we have

$$(19) \quad \|f\|_{L_M} \leq 1 \Rightarrow \int_{\mathbb{T}} M(|f|) \leq \|f\|_{L_M}$$

$$(20) \quad \|f\|_{L_M} \geq 1 \Rightarrow \int_{\mathbb{T}} M(|f|) \geq \|f\|_{L_M}.$$

2) From this fact we may deduce, that

$$(21) \quad c_1 \left(1 + \int_{\mathbb{T}} M(|f|) \right) \leq \|f\|_{L_M} \leq c_1 \left(1 + \int_{\mathbb{T}} M(|f|) \right)$$

provided $\|f\|_{L_M} = 1$.

3) For any measurable set E we have

$$\|\mathbb{I}_E\|_{L_M} = o(1) \text{ as } |E| \rightarrow 0 \text{ ([13], p. 89).}$$

4) If M satisfies Δ_2 -condition, that is

$$M(2t) \rightarrow cM(t), t > t_0,$$

then the set of all kind of polynomials is dense in L_M ([13], p. 99).

5) From (19) it follows that for any sequence of function f_n the condition $\|f_n\|_{L_M} \rightarrow 0$ implies $\int_{\mathbb{T}} M(|f_n|) \rightarrow 0$.

Proof of Theorem 2. We will deal with two M -functions

$$\Phi(t) = t \log^+ t,$$

$$\Psi(t) = \exp t - 1.$$

Combining (21) with Theorem 1, we may obtain

$$(22) \quad |\{(x, y) \in \mathbb{T}^2 : \text{BMO}[S_{nn}(x, y, f)] > \lambda\}| \lesssim \frac{\|f\|_{L_\Phi}}{\lambda}.$$

Indeed, at first we deduce the case when $\|f\|_{L_\Phi} = 1$, then, using a linearity principle, we get the inequality in the general case.

The inequality

$$(23) \quad \|f\|_{L_\Psi} \lesssim \|f\|_{\text{BMO}},$$

proved in [20]. It is an immediate consequence of the John-Nirenberg theorem. Denote

$$(24) \quad \mathcal{B}f(x, y) = \sup_{0 \leq n < \infty} \left\| \sum_{k=0}^n S_{kk}(x, y, f) \mathbb{I}_{\delta_k^n}(t) \right\|_{L_\Psi(dt)}.$$

Notice, that by the definition we have

$$\text{BMO}[S_{nn}(f, x, y)] = \sup_{0 \leq n < \infty} \left\| \sum_{k=0}^n S_{kk}(x, y, f) \mathbb{I}_{\delta_k^n}(t) \right\|_{\text{BMO}(dt)}.$$

So, taking into account (22) and (23) we obtain

$$(25) \quad |\{(x, y) \in \mathbb{T}^2 : \mathcal{B}f(x, y) > \lambda\}| \lesssim \frac{\|f\|_{L_\Phi}}{\lambda}.$$

On the other hand we have

$$\begin{aligned} \frac{1}{n+1} \sum_{k=0}^n (\exp A|S_{kk}(x, y, f) - f(x, y)| - 1) \\ = \frac{1}{n+1} \sum_{k=0}^n \Psi(A|S_{kk}(x, y, f) - f(x, y)|) \\ = \int_0^1 \Psi \left(A \sum_{k=0}^n |S_{kk}(x, y, f) - f(x, y)| \mathbb{I}_{\delta_k^n}(t) \right) dt. \end{aligned}$$

Thus, according the property 5) of Orlicz spaces, to prove the theorem it is enough to prove that

$$(26) \quad \left\| \sum_{k=0}^n (S_{kk}(x, y, f) - f(x, y)) \mathbb{I}_{\delta_k^n}(t) \right\|_{L_\Psi(dt)} \rightarrow 0,$$

almost everywhere on \mathbb{T}^2 as $n \rightarrow \infty$, for any $f \in L_\Phi$. It is easy to observe, that (26) holds if f is a real trigonometric polynomial in two variables. Indeed, if $P(x, y)$ is a polynomial of degree m , then we have

$$S_{kk}(x, y, P) - P(x, y) \equiv 0, \quad k \geq m.$$

Therefore, if $n \geq m$, then we get

$$\left| \sum_{k=0}^n (S_{kk}(x, y, P) - P(x, y)) \mathbb{I}_{\delta_k^n}(t) \right| \leq C \cdot \mathbb{I}_{[0, m/(n+1)]}(t),$$

where C is a constant, depending on P . Then, applying the property 3) of Orlicz spaces, we conclude that (26) holds for if $f = P$. To prove the general case, we consider the set

$$(27) \quad G_\lambda = \{(x, y) \in \mathbb{T}^2 : \limsup_{n \rightarrow \infty} \left\| \sum_{k=0}^n (S_{kk}(x, y, f) - f(x, y)) \mathbb{I}_{\delta_k^n}(t) \right\|_{L_\Psi(dt)} > \lambda\}.$$

To complete the proof of theorem, it enough to prove that $|G_\lambda| = 0$ if $\lambda > 0$. It is easy to check that $\Phi(t)$ satisfies the Δ_2 -condition. Therefore, according the property 4), we may chose a polynomial $P(x, y)$ such that

$$\|f - P\|_{L_\Phi} < \varepsilon.$$

Then, using (25) and Chebishev's inequality, for any $\lambda > 0$ we get

$$\begin{aligned}
|G_\lambda| &= |\{(x, y) \in \mathbb{T}^2 : \\
&\limsup_{n \rightarrow \infty} \left\| \sum_{k=0}^n (S_{kk}(x, y, f - P) - f(x, y) + P(x, y)) \mathbb{I}_{\delta_k^n}(dt) \right\|_{L_\Psi(dt)} > \lambda\}| \\
&\leq |\{\mathcal{B}(f - P)(x, y) + |f(x, y) - P(x, y)| > \lambda\}| \\
&\lesssim \left(\frac{1}{\lambda} + \frac{1}{\Phi(\lambda)} \right) \|f - P\|_{L_\Phi} < c(\lambda)\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ may be taken sufficiently small, we conclude $|G_\lambda| = 0$ if $\lambda > 0$. \square

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